Particle Creation in Kaluza-Klein Cosmology

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ABSTRACT

We exactly calculate the particle number N of scalar fields which are created from an initial vacuum in certain higher-dimensional cosmological models. The spacetimes in these models are the four-dimensional Chitre-Hartle or radiation-dominated universe with extra spaces which are static or power-law contracting. Except for some models in which no particles could be produced, the distribution of created particles shows a thermal behavior, at least in the limit of high three-dimensional "momentum" k. In some models, N does not depend on the magnitude of the extra-dimensional "momentum" k_c if k_c is nonvanishing. A cutoff momentum k_c may emerge in some models, and particles with $k \leq k_c$ could not be produced. We also discuss these results.

1 Introduction

Particle production on the background geometry of cosmology has been studied by several authors [1,2]. To determine the number of produced particles one must calculate the overlap function between the initial and final state of the particle, which is induced by the dynamical changing of the universe. Many exactly solvable models have been found for theories in a 3-flat Robertson-Walker universe [2]. However, it is difficult to solve exactly anisotropic models. To our knowledge, only the exact solution for the Kasner-like model in three dimensions has been obtained [3]. This then led Birrell and Davies [4] (see also ret. [5]) to develop a perturbation method for solving it. As the method they exploited could only treat models with small anisotropies, the problem of particle production in "realistic" higher- dimensional unified theories, e.g. Kaluza-Klein supergravity [6], superstring theory [7], etc., in which three spaces are expanding while the other extra spaces are contracting, could not be studied. Thus it is useful to evaluate exactly the number of particles produced on a Kaluza-Klein cosmology.

In this paper we will calculate exactly the particle number of scalar fields which are created from an initial vacuum in certain Kaluza-Klein cosmologies. The spacetimes in these models are the four-dimensional Chitre-Hartle or radiation-dominated universe with extra spaces which are static or power-law contracting. The results show that no particle could be created from some spacetimes even though they are not conformally flat. In some models, N does not depend on the magnitude of the extra-dimensional "momentum" k if $k \neq 0$, and no particle could be produced in the modes in which the three-dimensional momentum $k \leq k_c$. (The cutoff momentum k_c depends on the model.) We also find that the distribution of created particles does show a thermal spectrum at sufficiently short wavelengths, in agreement with conventional belief [8,9].

In section 2 we briefly derive the formulas and we use them to evaluate the number of particles created in some five-dimensional models in section 3. The extension to higher-dimensional theories and some discussions are given in section 4.

2   Cosmological Particle Creation: Formulation

The models considered in this paper are a scalar field Ψ minimally coupled to a five-dimensional spacetime with metric $g_{\mu\nu}$. The scalar field will satisfy the wave equation

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\Psi) + m^2\Psi = 0. \tag{2.1}$$

The line element we used is

$$ds^2 = dt^2 - A^2(t)d\mathbf{x}^2 - B^2(t)dx_5^2, \tag{2.2}$$

where dx_5 is the coordinate of the fifth space. Introducing the so-called conformal time η defined by

$$d\eta = A(t)^{-1}dt, \tag{2.3}$$

we write eq. (2.2) as

$$ds^2 = A^2(t)(d\eta^2 - d\mathbf{x}^2) - B^2(\eta)dx_5^2. \tag{2.4}$$

We now expand Ψ in terms of a complete set of modes $u_{\mathbf{k},k_5}$,

$$\Psi(\mathbf{x}, x_5, \eta) = \sum_{\mathbf{k},k_5} a_{\mathbf{k},k_5}u_{\mathbf{k},k_5}(\mathbf{x}, x_5, \eta) + a_{\mathbf{k},k_5}^\dagger u_{\mathbf{k},k_5}^*(\mathbf{x}, x_5, \eta), \tag{2.5}$$

where the creation operators $a_{\mathbf{k},k_5}^\dagger$, and annihilation operators $a_{\mathbf{k},k_5}$ satisfy the usual commutation relations. After defining

$$u_{\mathbf{k},k_5} = S^{-1}(\eta)exp[-i(k\dot{x} + k_5x_5)]\Psi_{\mathbf{k},k_5}(\eta), \tag{2.6}$$

where

$$S(\eta) = A(\eta)B^{1/2}(\eta), \tag{2.7}$$

one can from eq. (2.1) obtain an ordinary differential equation for Ψ_l (l stands for $\mathbf{k}, k_5$),

$$\frac{d^2\Psi_l}{d\eta^2} + \Omega^2(\eta)\Psi_l(\eta) = 0, \tag{2.8}$$

where

$$\Omega^2(\eta) = k^2 + A(\eta)[k_5^2/B(\eta)^2 + m^2] - \frac{d^2 S/d\eta^2}{S}. \tag{2.9}$$

Suppose the spacetime to be static at $\eta \to \pm\infty$, or so slowly varying that an adiabatic definition of the vacuum state makes sense. We can then use the WKB approximation to find the solutions of eq. (2.8) in the limits $\eta \to \pm\infty$. Taking this as the "asymptotical condition" we in general have two solutions Ψ_l^{in} and Ψ_l^{out} in eq. (2.8). As both sets are complete, we can expand Ψ_l^{out} in terms of Ψ_l^{in},

$$\Psi_l^{out} = \alpha_l \Psi_l^{in} + \beta_l \Psi_l^{in}*, \tag{2.10}$$

and the vacuum of the Ψ_l^{out} mode will contain $|\beta_l|^2$ particles in the $\Psi^i n_l$ mode [1,2]. Thus the number of produced particles N_l is

$$N_i = |\beta_l|^2. \tag{2.11}$$

Note that the Wronskian condition which guarantees the conservation of particle flux and the unitarity of the S matrix implies

$$|\alpha_l|^2 - |\beta_l|^2 = 1. \tag{2.12}$$

The remaining work is to find the exact solution ofeq. (2.8) for a given metric, which in turn exactly determines the value of β_l and thus the number N_l. Before doing this, however, we

will briefly discuss two classes of second order differential equations which correspond to the wave equations (2.8) for the models analyzed in the next section.

$$(a) \quad d^2\Psi/d\eta^2 + (Fe^{\lambda\eta} + G)\Psi = 0. \quad \lambda > 0. \tag{2.13}$$

The general solutions can be expressed in terms of the Bessel function,

$$\Psi(\eta) = CZ_\nu(X), \quad X = \frac{2\sqrt{F}}{\lambda}e^{\lambda\eta/2}, \tag{2.14}$$

where C is a normalization constant and ν is the order of the Bessel function,

$$\nu = -\frac{2i\sqrt{G}}{\lambda}. \tag{2.15}$$

At the initial time, $t \to -\infty$, Ω^2 approachs the constant G, and the state is static. On the other hand, at the final time, $t \to +\infty$, Ω^2 does not become a constant number and the physical state is not static. However, as the measure of adiabaticity $\Omega^2 d\Omega/d\eta \to 0$, we can use the WKB method to define the adiabatic vacuum. Using these as the asymptotical conditions we can find the in-mode and out-mode solutions:

$$\Psi_l^{in} \propto J_\nu(X), \tag{2.16a}$$

$$\Psi_l^{out} \propto H_\nu^{(2)}(X), \tag{2.16b}$$

where $J_\nu(X)$ and $H_\nu^{(2)}(X)$ are the Bessel function of the first class and the Hankel function, respectively. Note that we cannot choose the initial state at $\eta = 0$, as the measure of adibaticity at $\eta = 0$ is not zero and the adiabatic definition of particles does not make sense.

Using the relation

$$H_\nu(X) = -\frac{i}{sin\pi\nu}[e^{i\pi\nu}J_\nu(X) - J_{-\nu}(X)], \tag{2.17}$$

one finally obtains the number of produced particles,

$$N = \frac{1}{|e^{2i\pi\nu}| - 1}. \tag{2.18}$$

Note that if ν is a real number then no panicles will be produced.

$$(b) \quad d^2\Psi/d\eta^2 + (F\eta^{-2} + G)\Psi = 0. \tag{2.19}$$

This case will have asymptotically static states if the universe begins at $\eta = \pm\infty$ and ends at $\eta = \mp\infty$. The solutions are

$$\Psi^{in} = \Psi^{out} \propto \sqrt{\eta} \; H_\nu^{(2)}(\sqrt{G}\eta), \quad \nu = 1/4 - F, \tag{2.20}$$

and thus, trivially, no particles have been created.

3 Cosmological Particle Creation: Models and Results

We now present the exact number of particles produced in the following pacetimes:

Model 1. $A(\eta) = A_0 e^{\alpha\eta}$, $B(\eta) = B_0$. This spacetime describes the four-dimensional Chitre-Hartle universe [10] with a static extra space. (Note that the Chitre-Hartle metric is a solution of the Einstein equation modified by including a one-loop quantum-gravitational correction arising from the quantum trace anomaly [11].) The wave equation to be solved is eq. (2.13) with

$$F = A_0^2(k_5^2 B_0^{-2} + m^2), \quad G = k^2 - \alpha^2, \quad \lambda = 2\alpha. \tag{3.1}$$

Thus the number of produced particles is

$$N_{\mathbf{x},x_5} = \left[exp[(2\pi/alpha)\sqrt{k^2 - \alpha^2}] - 1\right]^{-1}. \tag{3.2}$$

It is independent of the fifth dimensional "momentum" k_5 and particle mass m, and shows a black-body spectrum when $k >> \alpha$. Note that particles with mode $k \leq \alpha$ could not be created. One must also notice in the case $m = k_5 = 0$, that $F = 0$ and eq. (2.13) admits a plane wave solution, therefore, no particle could be created.

The extra space in this model is static, while that in the next model is contracting.

Model2. $A(\eta) = A_0 \, exp^{\alpha\eta}$, $B(\eta) = B_0 \, exp^{-\beta\eta}$. This space-time describes the four-dimensional Chitre-Hartle metric with a power-law contracting extra space if $\beta > 0$. When $m = 0$ then the wave equation to be solved is eq. (2.13) with

$$F = \frac{A_0^2}{B_0^2}k_5^2, \quad , G = k^2 - (\alpha - 1/2\beta)^2, \quad \lambda = 2(\alpha + \beta). \tag{3.3}$$

Thus the number of produced particles is

$$N_{\mathbf{x},x_5} = \left[exp[(2\pi/(\alpha + \beta)\sqrt{k^2 - (\alpha - 1/2\beta)^2}] - 1\right]^{-1}. \tag{3.4}$$

It is independent of the fifth dimensional "momentum" k_5, and shows a black-body spectrum when $k >> |\alpha - 1/2\beta|$. Note that no particle with mode $k \leq |\alpha - 1/2\beta|$ could be produced. One must also notice if $k_5 = 0$, that $F = 0$ and eq. (2.13) admits a plane wave solution, therefore, no particle could be produced.

Model 3. $A(\eta) = A_0\eta$, $B(\eta) = B_0$. This spacetime describes the radiative Robertson-Walker universe with a static extra space. The wave equation to be solved is

$$d^2\Psi/d\eta^2 + [k^2 + A_0^2(m^2 + k_5^2/B_0^2)]\Psi = 0. \tag{3.5}$$

This equation has been analyzed by Audretsch and Schafer [12], after replacing $m^2 + k_5^2/B_0^2$ by m^2. As discussed in that paper, in order to have the adi- abatic region in which the ingoing and outgoing particle definition makes sense without changing the Einstein equation for all time, the universe is completed in passing through the singularity by a time-symmetric

one. Then the wave equation could be solved by the parabolic cylinder function, and the number of produced particles is

$$N_{\mathbf{x},x_5} = exp\left(-\pi k/A_0\sqrt{k_5^2 B_0^{-2} + m^2}\right). \tag{3.6}$$

Our model is only a simple extension of that studied in ref. [12], and the result also shows a non-relativistic thermal spectrum.

Model 4. $A(\eta) = A_0\eta$, $B(\eta) = B_0\eta^{-2C}$. This spacetime describes the four-dimensional radiative universe with a contracting extra space if $C > 0$. For the case $C = 1/2$, it just describes the five-dimensional Kasner universe. If $m = k_5 = 0$ then the wave equation to be solved is eq. (2.19) with

$$F = C(C-1), \quad G = k^2, \tag{3.7}$$

and no particle will be produced eventually. Note that when $\eta \to 0$, $B(\eta) \to \infty$, so during the evolution the extra space will be infinitely large. To see that the production of no particles is not always the consequence of having an infinite space, we consider the next model in which the exact solution could be found even for the non-zero ks mode. (However, the extra space is expanding.)

Model 5. $A(\eta) = A_0\eta$, $B(\eta) = B_0\eta^2$. If $m = 0$ then the wave equation to be solved is just eq. (2.19) with

$$F = A_0^2 B_0^{-2} k_5^2, \quad G = k^2 - 2, \tag{3.8}$$

and no particles will be produced also.

Model 6. For any model whose $S(\eta)(= A(\eta)B^{1/2}(\eta))$ is a power function of η, in the case $k_5 = m = 0$ the wave equation becomes as eq. (2.19), and thus no particle could be created.

4 Discussions

Although all the above models are described in 1+3+1 dimensions, the extension to higher dimension is straightforward. Let us consider (1 +3+M)-dimensional theories in which the scale functions of the three dimensions and extra M dimensions are denoted as $A(\eta))$ and $B(\eta)$, respectively. Then the wave equation ofeq. (2.8) could still be used if the $S(\eta)$ function is now defined by

$$S(\eta) = A(\eta)B(\eta)^{M/2}. \tag{4.1}$$

One can see that all the above results will be unchanged for theories in any other dimension.

Let us discuss our results and address some questions arisen from them.

(1) In models 1, 2 and 3, the particle spectrum does show a thermal behavior, as a consequence of the general property argued by Parker [8] and shown recently by Kandrup [9J.

(2) A cutoff momentum is displayed in models 1 and 2, below which no particle will be produced. This is because below the threshold the G term in eq. (2.13) is negative, and at initial time, $\eta \to -\infty$, Ω^2 becomes a negative number. This implies that the wave function Ψ is of exponential type, and the particle cannot propagate in the space. Thus no particle will be created.

(3) For the massless case of models 1 and 2, one could assert that particle creation is independent of the fifth dimensional momentum k_5 only if k_5 is non-vanishing. A "thorough" independence of k_5 shows itself only in the massive case in model 1. A "physical explanation" for this "independence" is unknown now.

(4) Intuitively, particles of higher mass are produced in smaller quantity, so the fact that the particle spectrum in model 1 does not depend on the particle mass (excluding the case k_5, $m = 0$, in which no particle is produced) is strange. This needs fur- ther explanation.

(5) In models 4, 5 and 6, no particles are created, although these spacetimes are not conformally flat. Is there a simple argument against particle production in these models ? This remains to be found out.

(6) Furthermore, according to the intuition that the entropy of a particle field should be directly connected with the number of quanta in the field, how will the Kandrup-Hu entropy [13,14] be changed in these no-particle-production models? Further analyses are needed to answer this question.

Work on these problems is in progress.

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